The Reciprocal Pairing: A Symmetry-Based Definition of the Product of All Elements in Division Algebras

$$\prod_{a \neq 0} a = e^{i\pi}$$

Joerg Moesch

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The Reciprocal Pairing: A Symmetry-Based Definition of the Product of All Elements in Division Algebras

Joerg Moesch
mailtojoerg@proton.me

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Abstract

This paper introduces the **reciprocal pairing**, a structural method for determining the product of all non-zero elements of a division algebra—a product that is not well-defined in classical analysis. The method pairs each element with its reciprocal under the natural involution $\iota(x) = x^{-1}$ and thereby reduces the problem to computing the product of the fixed points $\mathrm{Fix}(\iota) = \{x \mid x^2 = 1\}$.

Applied to the four real normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, this yields the canonical value -1 in each case. The procedure is purely algebraic, order-independent, and requires no analytic tools.

Keywords: Reciprocal pairing, algebraic symmetry, division algebras, involution, fixed points, infinite products

MSC2020: 16K20, 17A35

1 Introduction

"What is the multiplicative combination of everything?" In classical analysis, the product over all non-zero elements of an uncountable set is not well-defined. Analytic regularization methods such as zeta-function regularization [5] are fundamentally unsuitable for products over unordered or continuous sets like $\mathbb{R}^\times$, since they require both analytic continuation and a linear ordering of the factors.

Building on an earlier idea of the author [4, p. 60] and by analogy to the known generalization of Wilson's theorem to finite fields [2, p. 162], this paper proposes a **structural approach** that respects the inherent algebraic symmetry of multiplication. For a division algebra K, the involution $x \mapsto x^{-1}$ on its multiplicative group $K^\times$ [3] is a canonical symmetry that follows from the group structure alone; it requires no additional algebraic structure (such as conjugation, norm, or ordering) and exists in every group of invertible elements. The reciprocal pairing uses this symmetry to define the product over all non-zero elements of a division algebra K without analytic tools.

2 The Reciprocal Pairing

Definition 2.1 (Reciprocal pairing)**.** Let K be a division algebra with multiplicative group $K^\times = K \setminus \{0\}$. The *reciprocal involution*

$$\iota \colon K^\times \to K^\times, \quad \iota(x) = x^{-1}$$

induces the disjoint orbit decomposition of $K^\times$ under the group $\langle \iota \rangle = \{\mathrm{id}, \iota\}$ generated by ι:

$$K^\times = \mathrm{Fix}(\iota) \sqcup \bigsqcup_{\{x,x^{-1}\}\in\mathcal{P}} \{x, x^{-1}\}$$

where

- $\mathrm{Fix}(\iota) = \{x \in K^\times \mid x^2 = 1\}$ are the fixed points,
- $\mathcal{P} = \{\{x, x^{-1}\} : x \in K^\times, x \neq x^{-1}\}$ is the set of (unordered) two-element orbits.

The **product** over all non-zero elements of a division algebra is defined via the **reciprocal pairing** as:

$$\boxed{\prod_{x\in K^\times} x := \prod_{z\in\mathrm{Fix}(\iota)} z}$$

Theorem 2.2 (Orbit contribution)**.** *For each pair $\{x, x^{-1}\}$ with $x \notin \mathrm{Fix}(\iota)$ we have:*

$$x \cdot \iota(x) = x \cdot x^{-1} = 1$$

Since all two-element orbits $\{x, x^{-1}\}$ contribute multiplicatively by the neutral factor 1, only the fixed points contribute to the total product according to Definition 2.1.

Proof. In division algebras, for every $x \neq 0$ the multiplicative inverse x^{-1} is uniquely determined by the condition

$$x \cdot x^{-1} = 1$$

where 1 denotes the unit element of the algebra. For $x \notin \mathrm{Fix}(\iota)$ we additionally have $x^{-1} \neq x$, so that the pairs $\{x, x^{-1}\}$ are well-defined and each yields $x \cdot x^{-1} = 1$. □

3 Application to Real Normed Division Algebras

In what follows we consider the four real normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [1, pp. 4–5].

Theorem 3.1 (Fixed points in normed division algebras)**.** *For $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ we have:*

$$\mathrm{Fix}(\iota) = \{x \in K^\times \mid x^2 = 1\} = \{1, -1\}$$

Proof. Let $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $x \in K$ with $x^2 = 1$. Every element can be written as $x = a+y$, where:

- $a \in \mathbb{R}$ is the real part,
- y is purely imaginary ($\overline{y} = -y$) with $y^2 = -|y|^2 \leq 0$.

Then:

$$x^2 = (a + y)^2 = a^2 + 2ay + y^2 = a^2 - |y|^2 + 2ay = 1.$$

Since $a^2 - |y|^2 \in \mathbb{R}$ and $2ay$ is purely imaginary, separation yields:

$$\begin{cases} a^2 - |y|^2 = 1 \\ 2ay = 0 \end{cases}$$

- If $a = 0$: then $0 - |y|^2 = 1 \Rightarrow |y|^2 = -1$, impossible.
- If $y = 0$: then $x = a \in \mathbb{R}$ with $a^2 = 1$, hence $a = \pm 1$.

Thus 1 and -1 are the only solutions of $x^2 = 1$ in all four algebras. □

Theorem 3.2 (Main result)**.** *For every real normed division algebra we have:*

$$\boxed{\prod_{x \in K^\times} x = -1}$$

Proof. By Theorem 3.1, $\mathrm{Fix}(\iota) = \{1, -1\}$ for all $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. For $x \notin \mathrm{Fix}(\iota)$, Theorem 2.2 gives $x \cdot x^{-1} = 1$ (also in $\mathbb{H}$ and $\mathbb{O}$, since $x^{-1} = \overline{x}/|x|^2$ with $x\overline{x} = \overline{x}x$). Since the definition only involves products of two elements (namely x and x^{-1}), which are well-defined even in non-associative algebras, and a finite product over $\mathrm{Fix}(\iota)$, no associativity is required. Because 1 and -1 lie in the center of every real division algebra, their product is independent of order. Using Definition 2.1 we therefore obtain:

$$\prod_{x \in K^\times} x = \prod_{z \in \{1,-1\}} z = 1 \cdot (-1) = -1.$$

□

The concrete results for the four real division algebras are summarized in Table 1.

Table 1: The total product $\displaystyle\prod_{x \in K^\times} x$ for real normed division algebras

Algebra	$\mathrm{Fix}(\iota)$	$\displaystyle\prod_{x \in K^\times} x$
$\mathbb{R}^\times$	$\{1, -1\}$	-1
$\mathbb{C}^\times$	$\{1, -1\}$	-1
$\mathbb{H}^\times$	$\{1, -1\}$	-1
$\mathbb{O}^\times$	$\{1, -1\}$	-1

4 Further Applications

The reciprocal pairing is not limited to the classical real division algebras. Its purely algebraic nature allows transfer to any multiplicative structure that possesses a well-defined inversion and a finite set of fixed points under the involution $x \mapsto x^{-1}$. Possible application contexts include:

- **p-adic fields**: For the multiplicative groups $\mathbb{Q}_p^\times$ of p-adic numbers or finite extensions thereof, which mix discrete and continuous structures, the principle could lead to new arithmetic invariants.
- **Lie groups**: For compact Lie groups, which exhibit both continuous and algebraic structures, the reciprocal pairing could provide new perspectives on invariant integration.
- **Physical path integrals**: In quantum field theories with quaternionic or octonionic fields, the reciprocal pairing might offer a purely algebraic regularization of functional determinants, presenting an alternative to the established zeta-regularization [5].
- **Geometric machine learning**: In neural networks with weights in $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, the principle could be used to define symmetry-preserving regularization terms.

A systematic investigation of these directions is left for future work.

Concluding Remarks

The seemingly naive question "What is the multiplicative combination of everything?" has led to the *reciprocal pairing* — a structural definition that is not based on analytic limits, but on the internal symmetry of multiplication. By exploiting the involution $\iota(x) = x^{-1}$ [3], the problem is reduced to computing the product of the fixed points.

In the finite case, this principle reproduces the known generalization of Wilson's theorem to finite fields, $\prod_{x \in \mathbb{F}_q^\times} x = -1$ [2, p. 162]. In the infinite case, it leads consistently to the same result -1 for all four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

The consistency of the result -1 underscores the universality of the symmetry-based approach and opens perspectives for further investigations in algebra and mathematical physics.

Transparency Note

This paper was created with the assistance of DeepSeek AI. The AI supported mathematical precision, LaTeX implementation, and translation from German to English. The mathematical ideas, results, and conclusions originate entirely from the author.

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Correspondence:
Joerg Moesch
c/o Block Services, Stuttgarter Straße 106, D-70736 Fellbach
mailtojoerg@proton.me